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Factorization in finite quantum systems

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Abstract

Unitary transformations in an angular momentum Hilbert space $H(2j+1)$, are considered. They are expressed as a finite sum of the displacement operators (which play the role of $SU(2j+1)$ generators) with the Weyl function as coefficients. The Chinese remainder theorem is used to factorize large qudits in the Hilbert space $H(2j+1)$ in terms of smaller qudits in Hilbert spaces $H(2j_i+1)$. All unitary transformations on large qudits can be performed through appropriate unitary transformations on the smaller qudits.

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1. Introduction

Much of the work on quantum computation has been based on qubits in two-dimensional Hilbert spaces. More recently the use of multi-dimensional Hilbert spaces (qudits) as a potentially more powerful tool for quantum computation has been studied [1]. At the same time, the physical implementation of the required unitary transformations is much more difficult in Hilbert spaces with large dimension. $SU(d)$ transformations in a d-dimensional Hilbert space have $d^2 - 1$ generators, which for large d might be very difficult to implement practically.

In this paper we show how we can factorize large qudits in terms of smaller ones, so that all unitary transformations on the large qudits can be performed through appropriate unitary transformations on the smaller qudits. This factorization is similar to that used in a classical context in 'fast Fourier transforms' where a Fourier transform in a large Hilbert space is reduced to many Fourier transforms in smaller Hilbert spaces which are appropriately combined to give the result for the large Hilbert space.

In section 2 we present briefly the quantum mechanics in a $(2j+1)$-dimensional angular momentum system with integer j (Bose case) and introduce the notation. We explain that the displacement operators can be used as generators of $SU(2j+1)$ transformations; and we show that finite $SU(2j+1)$ transformations can be expressed as a finite sum of the displacement operators, with the Weyl function as coefficients.

In section 3 we use the Chinese remainder theorem to factorize large qudits in terms of smaller ones. We show that all unitary transformations on large qudits can be performed through appropriate unitary transformations on the smaller qudits. We conclude in section 4 with a discussion of our results.

2. Qudits

Finite quantum systems have been studied originally by Weyl and Schwinger [2]. More recently this work has been applied in various contexts by various authors [3]. In [4] we have applied these ideas in the context of the angle-angular momentum quantum phase space. In this section we first introduce the notation and review briefly some of these ideas in the context of qudits. We then discuss the use of displacement operators as generators of $SU(2j+1)$ transformations and we express finite $SU(2j+1)$ transformations as finite sums of the displacement operators, with the Weyl function as coefficients.

We denote as $|J, jm\rangle$ the usual angular momentum states; m belongs to $\mathcal{Z}(2j+1)$ (the integers modulo $2j+1$). The states $|J, jm\rangle$ span the Hilbert space $H(2j+1)$. The finite Fourier transform is defined as

$$F = (2j+1)^{-1/2} \sum_{m,n} \omega(mn)|J; jm\rangle\langle J; jn| \tag{1}$$

$$\omega(\alpha) = \exp\left[\mathrm{i}\frac{2\pi\alpha}{2j+1}\right] \qquad FF^\dagger = F^\dagger F = \mathbf{1} \qquad F^4 = \mathbf{1}. \tag{2}$$

Using these Fourier transforms we have introduced [4] the θ-basis of angle states $|\theta; jm\rangle$ as follows:

$$|\theta; jm\rangle = F|J; jm\rangle = (2j+1)^{-1/2} \sum_{n} \omega(mn)|J; jn\rangle. \tag{3}$$

We have also introduced the angle operators $\theta_z = FJ_zF^\dagger$, $\theta_+ = FJ_+F^\dagger$, $\theta_- = FJ_-F^\dagger$, which obey the $SU(2)$ algebra. The displacement operators are defined as

$$X = \exp\left[-\mathrm{i}\frac{2\pi}{2j+1}\theta_z\right] \qquad Z = \exp\left[\mathrm{i}\frac{2\pi}{2j+1}J_z\right] \tag{4}$$

$$X^{2j+1} = Z^{2j+1} = \mathbf{1} \qquad X^\beta Z^\alpha = Z^\alpha X^\beta \omega(-\alpha\beta) \tag{5}$$

where α, β are integers in $\mathcal{Z}(2j+1)$, and perform displacements along the J_z and θ_z axes, as follows:

$$X^\beta|J; jm\rangle = |J; jm+\beta\rangle \qquad X^\beta|\theta; jm\rangle = \omega(-\beta m)|\theta; jm\rangle \tag{6}$$

$$Z^\alpha|J; jm\rangle = \omega(m\alpha)|J; jm\rangle \qquad Z^\alpha|\theta; jm\rangle = |\theta; jm+\alpha\rangle. \tag{7}$$

The general displacement operators are defined as

$$D(\alpha, \beta) = Z^\alpha X^\beta \omega(-2^{-1}\alpha\beta) \qquad [D(\alpha, \beta)]^\dagger = D(-\alpha, -\beta). \tag{8}$$

Many of the formulae below are proved with the use of the relations

$$\begin{aligned} &\langle J; jn|D(\alpha, \beta)|J; jm\rangle = \delta(n, m+\beta)\omega(2^{-1}\alpha\beta + \alpha m) \\ &\sum_{m=-j}^{j} \omega(m\alpha) = (2j+1)\delta(\alpha, 0) \end{aligned} \tag{9}$$

where $\delta(n, m)$ is the Kronecker delta which is equal to 1 when $n = m (\mathrm{mod}\,(2j+1))$.

We also define the parity operator around the origin

$$P(0,0) = F^2 = \sum_{m=-j}^{j} |J;\, j-m\rangle\langle J;\, jm| = \sum_{m=-j}^{j} |\theta;\, j-m\rangle\langle\theta;\, jm| \qquad (10)$$

and the displaced parity operator

$$P(\alpha,\beta) = D(\alpha,\beta)P(0,0)[D(\alpha,\beta)]^\dagger = D(2\alpha,2\beta)P(0,0) = P(0,0)[D(2\alpha,2\beta)]^\dagger \qquad (11)$$

$$P(\alpha,\beta)P(\gamma,\delta) = D(2\alpha-2\gamma, 2\beta-2\delta)\omega(2\beta\gamma-2\alpha\delta). \qquad (12)$$

Wigner and Weyl or characteristic functions (discussed in the harmonic oscillator context in [5–7]) can be defined for general operators, which are not necessarily density matrices. In our context we are interested in Wigner and Weyl functions corresponding to a unitary transformation U.

The Wigner function corresponding to an operator U, is defined in terms of the displaced parity operator as

$$W(U;\alpha,\beta) = \mathrm{Tr}[UP(\alpha,\beta)]. \qquad (13)$$

We note that since U is in general a non-Hermitian operator, the Wigner function is complex.

The Weyl function corresponding to an operator U, is defined in terms of the displaced operator as

$$\tilde{W}(U;\alpha,\beta) = \mathrm{Tr}[UD(\alpha,\beta)] = (2j+1)^{-1}\sum_{\gamma,\delta} W(U;\gamma,\delta)\omega(\alpha\delta-\beta\gamma). \qquad (14)$$

It is seen that it is related to the Wigner function through a 'two-dimensional' Fourier transform (indicated with the tilde in the notation). For later purposes, we prove the important formula

$$U = (2j+1)^{-1}\sum_{\alpha,\beta} W(U;\alpha,\beta)P(\alpha,\beta) = (2j+1)^{-1}\sum_{\alpha,\beta}\tilde{W}(U;-\alpha,-\beta)D(\alpha,\beta). \qquad (15)$$

It can be proved if we take the matrix elements of both sides with regard to the states $\langle J;\, jn|$ and $|J;\, jm\rangle$ and use equations (9).

2.1. *Infinitesimal transformations*

The $(2j+1)^2 - 1$ displacement operators (with $(\alpha,\beta) \neq (0,0)$) are generators for the $SU(2j+1)$ transformations in the Hilbert space $H(2j+1)$ [8]. They are an alternative to the usual Cartan–Weyl generators. Their commutator is

$$\begin{aligned}[D(\alpha_1,\beta_1), D(\alpha_2,\beta_2)] &\equiv D(\alpha_1,\beta_1)D(\alpha_2,\beta_2) - D(\alpha_2,\beta_2)D(\alpha_1,\beta_1)\\ &= 2\mathrm{i}\sin\left[\frac{2\pi}{2j+1}2^{-1}(\alpha_1\beta_2-\alpha_2\beta_1)\right]D(\alpha_1+\alpha_2,\beta_1+\beta_2). \qquad (16)\end{aligned}$$

Therefore infinitesimal $SU(2j+1)$ transformations can be written as

$$g = \mathbf{1} + \sum_{\alpha,\beta}\lambda(\alpha,\beta)D(\alpha,\beta) \qquad \lambda(\alpha,\beta) + [\lambda(-\alpha,-\beta)]^* = 0 \qquad (17)$$

where $\lambda(\alpha,\beta)$ are infinitesimal coefficients, subject to the above 'unitarity constraint'.

2.2. Finite transformations

Finite $SU(2j+1)$ transformations involve the exponentials of the generators. The generators are here finite matrices, and since the exponential of a finite matrix is a polynomial, we can write an arbitrary unitary operator U as

$$U = \sum_{\alpha,\beta} \mu(\alpha,\beta) D(\alpha,\beta) \qquad \mu(\alpha,\beta) = (2j+1)^{-1} \tilde{W}(U; -\alpha, -\beta) \tag{18}$$

where from equation (15) we see that the coefficients are the Weyl functions. Equation (18) shows that the displacement operators are the basic 'building blocks' for general unitary transformations of qudits. If we have 'black boxes' that perform the displacement transformations on qudits, then we can perform any unitary transformation using as coefficients the corresponding Weyl functions. For a product of two unitary operators U_1U_2, we can show that

$$\tilde{W}(U_1U_2; \alpha, \beta) = (2j+1)^{-1} \sum_{\alpha_1,\beta_1} \omega(2^{-1}\alpha_1\beta - 2^{-1}\alpha\beta_1) \times \tilde{W}(U_1; 2^{-1}\alpha + \alpha_1, 2^{-1}\beta + \beta_1) \tilde{W}(U_2; 2^{-1}\alpha - \alpha_1, 2^{-1}\beta - \beta_1). \tag{19}$$

This is proved if we take the matrix elements of both sides with regard to the states $\langle J; jn|$ and $|J; jm\rangle$ and use equations (9). We note that 2^{-1} in $\mathcal{Z}(2j+1)$ is equal to $j+1$.

Equation (15) also suggests the use of the displaced parity operators as an alternative set of building blocks for general unitary transformations of qudits:

$$U = \sum_{\alpha,\beta} \nu(\alpha,\beta) P(\alpha,\beta) \qquad \nu(\alpha,\beta) = (2j+1)^{-1} W(U; \alpha, \beta). \tag{20}$$

Here the coefficients are Wigner functions. We note that the displaced parity operators are *not* generators of $SU(2j+1)$ transformations; and in fact the product of two displaced parity operators is *not* a displaced parity operator (equation (12)). However, the Moyal star product (in the context of finite systems) tells us how to multiply two unitary operators U_1U_2 in this scheme. We show that

$$W(U_1U_2; \alpha, \beta) = (2j+1)^{-2} \sum_{\alpha_1,\beta_1,\alpha_2,\beta_2} \omega(2\alpha_2\beta_1 - 2\alpha_1\beta_2) \times W(U_1; \alpha + \alpha_1, \beta + \beta_1) W(U_2; \alpha + \alpha_2, \beta + \beta_2). \tag{21}$$

This is proved if we take the matrix elements of both sides with regard to the states $\langle J; jn|$ and $|J; jm\rangle$ and use equations (9). Taking three transformations, we can show that the Moyal product is associative.

3. Factorization of large qudits in terms of smaller ones

We consider the case where $2j+1$ can be factorized as $2j+1 = \prod_{i=1}^{N}(2j_i+1)$, where any two of the factors $2j_i+1$ are coprime. In this case we introduce an isomorphism between the Hilbert space $H(2j+1)$ and a product of Hilbert spaces $\prod_{i=1}^{N} H(2j_i+1)$. This isomorphism is based on the Chinese remainder theorem and it is similar to the prime factor scheme by Good in a classical context in 'fast Fourier transform' in order to reduce the computation time (e.g. [9]). The Chinese remainder theorem has also been used in quantum Fourier transforms (e.g. [10]).

In our context we show that unitary transformations on large qudits can be decomposed to several unitary transformations on smaller qudits; and appropriate combination of the results for the smaller qudits produces the unitary transformations of the large qudits. We have studied the isomorphism between $H(2j+1)$ and $\prod_{i=1}^{N} H(2j_i+1)$ in [11], and here we extend this work especially in the direction of how all unitary transformations on the large qudits (in $H(2j+1)$) can be decomposed into unitary transformations in the smaller qudits (i.e., in the various $H(2j_i+1)$).

3.1. Quantum states

We use the same notation as in [11], and introduce the integers

$$r_i = \frac{2j+1}{2j_i+1} \qquad t_i r_i = 1 \pmod{2j_i+1}. \tag{22}$$

The existence of t_i depends on the fact that the r_i and $2j_i+1$ are coprime. We also introduce the $s_i = t_i r_i$ in $\mathcal{Z}(2j+1)$. We note that since t_i is the inverse of r_i in $\mathcal{Z}(2j_i+1)$, the $s_i = t_i r_i$ defined in $\mathcal{Z}(2j+1)$ is an integer multiple of $(2j_i+1)$ plus 1. For a given m in $\mathcal{Z}(2j+1)$ we define the corresponding m_i and $\bar{m}_i$ in $\mathcal{Z}(2j_i+1)$ as follows:

$$\begin{aligned} &m_i = m \pmod{2j_i+1} \qquad \bar{m}_i = m t_i \pmod{2j_i+1} \\ &m = \sum_i m_i s_i = \sum_i \bar{m}_i r_i \pmod{2j+1}. \end{aligned} \tag{23}$$

We then have the one-to-one mappings $m \leftrightarrow \{m_i\} \leftrightarrow \{\bar{m}_i\}$. Using this we define a unitary isomorphism between the Hilbert space $H(2j+1)$ and the product $\prod H(2j_i+1)$ with

$$|J;\, jm\rangle \leftrightarrow \prod_{i=1}^{N} |J;\, j_i\bar{m}_i\rangle \qquad |\theta;\, jm\rangle \leftrightarrow \prod_{i=1}^{N} |\theta;\, j_i m_i\rangle. \tag{24}$$

The proof of this has been given in [11].

A general density matrix

$$\rho = \sum_{m,n} \sigma(m,n)|J;\, jm\rangle\langle J;\, jn| = \sum_{k,\ell} \tau(k,\ell)|\theta;\, jk\rangle\langle\theta;\, j\ell| \tag{25}$$

$$\sigma(m,n) = (2j+1)^{-1} \sum_{k,\ell} \tau(k,\ell)\omega(mk-\ell n) \tag{26}$$

can be written as

$$\rho = \sum_{\bar{m}_i,\bar{n}_i} \sigma(\{\bar{m}_i\},\{\bar{n}_i\}) \prod_{i=1}^{N}(|J;\, j_i\bar{m}_i\rangle\langle J;\, j_i\bar{n}_i|) = \sum_{k_i,\ell_i} \tau(\{k_i\},\{\ell_i\}) \prod_{i=1}^{N}(|\theta;\, j_i k_i\rangle\langle\theta;\, j_i\ell_i|) \tag{27}$$

$$\sigma(\{\bar{m}_i\},\{\bar{n}_i\}) = \left[\prod_{i=1}^{N}(2j_i+1)^{-1}\right] \sum_{k_i,\ell_i} \tau(\{k_i\},\{\ell_i\}) \prod_{i=1}^{N} \omega_i(\bar{m}_i k_i - \ell_i \bar{n}_i) \tag{28}$$

where the coefficients $\sigma(m,n)$ have been relabelled as $\sigma(\{\bar{m}_i\},\{\bar{n}_i\})$; and similarly for $\tau(k,\ell)$. The Fourier transform of equation (26) is equivalent to the Fourier transforms of equation (28). This can be proved with the use of equation (23) in conjuction with the fact that

$$\omega(r_i s_i) = \omega_i \equiv \exp\left(\mathrm{i}\frac{2\pi}{2j_i+1}\right) \qquad i \neq j \rightarrow \omega(r_i s_j) = 1. \tag{29}$$

3.2. Infinitesimal transformations

We first show that the displacement operators (generators of $SU(2j+1)$ transformations) in $H(2j+1)$ can be expressed as products of the displacement operators (generators of $SU(2j_i+1)$ transformations) in the various $H(2j_i+1)$,

$$D(\alpha,\beta)=\prod_{i=1}^{N}D_i(\alpha_i,\bar{\beta}_i) \tag{30}$$

where as explained in equation (23) $\alpha_i=\alpha(\text{mod } 2j_i+1)$ and $\bar{\beta}_i=\beta t_i\ (\text{mod } 2j_i+1)$. In order to prove this we first use equation (23) to prove that

$$\begin{aligned} Z&=\sum_m|\theta;\ jm+1\rangle\langle\theta;\ jm|=\sum_m\prod_{i=1}^{N}|\theta;\ j_im_i+1\rangle\langle\theta;\ j_im_i|=\prod_{i=1}^{N}Z_i\\ X&=\sum_m|J;\ jm+1\rangle\langle J;\ jm|=\sum_m\prod_{i=1}^{N}|J;\ j_i\bar{m}_i+t_i\rangle\langle J;\ j_i\bar{m}_i|=\prod_{i=1}^{N}X_i^{t_i}\end{aligned} \tag{31}$$

and then prove that

$$\left(\prod_{i=1}^{N}Z_i\right)^{\alpha}\left(\prod_{i=1}^{N}X_i^{t_i}\right)^{\beta}=\prod_{i=1}^{N}\left(Z_i^{\alpha_i}X_i^{\bar{\beta}_i}\right)\qquad \omega(-2^{-1}\alpha\beta)=\prod_{i=1}^{N}\omega_i(-2^{-1}\alpha_i\bar{\beta}_i). \tag{32}$$

We can also prove a relation analogous to (30) for the displaced parity operators. Using equations (11) and (30) we get

$$P(\alpha,\beta)=\prod_{i=1}^{N}P_i(\alpha_i,\bar{\beta}_i). \tag{33}$$

We next consider the displacement operators which act on only one of the Hilbert spaces $H(2j_i+1)$. They are those with $\alpha=s_i\alpha_i$ and $\beta=r_i\bar{\beta}_i$ where α_i and $\bar{\beta}_i$ are any integers in $\mathcal{Z}(2j_i+1)$. Indeed in this case we can easily show that for $j\neq i$ we have $\alpha_j=\bar{\beta}_j=0$; and consequently, the product on the right-hand side of equation (30) contains only one non-trivial factor

$$D(s_i\alpha_i,r_i\bar{\beta}_i)=D_i(\alpha_i,\bar{\beta}_i) \tag{34}$$

These displacement operators are the $\sum[(2j_i+1)^2-1]$ generators of the group $F=\prod SU(2j_i+1)$, which is a subgroup of $SU(2j+1)$. The group F describes factorizable transformations $U=\prod U_i$ which act independently on the various Hilbert spaces $H(2j_i+1)$. The rest $(2j+1)^2-1-\sum[(2j_i+1)^2-1]$ generators contain two or more non-trivial factors in the product on the right-hand side of equation (30) and produce non-factorizable transformations.

Infinitesimal $SU(2j+1)$ transformations can be written as

$$g=\mathbf{1}+\sum_{\alpha,\beta}\lambda(\alpha,\beta)D(\alpha,\beta)=\mathbf{1}+\sum_{\{\alpha_i\},\{\beta_i\}}\lambda(\{\alpha_i\},\{\bar{\beta}_i\})\prod_{i=1}^{N}D_i(\alpha_i,\bar{\beta}_i) \tag{35}$$

where $\lambda(\alpha,\beta)$ are infinitesimal coefficients. It is seen that all the unitary transformations of the large qudits can be constructed as combinations of unitary transformations of the smaller qudits. In fact, we only need $\sum[(2j_i+1)^2-1]$ 'black boxes' and then through equation (30) we can construct all $(2j+1)^2-1$ generators of $SU(2j+1)$ transformations.

3.3. Finite transformations

We next consider finite $SU(2j+1)$ transformations, and using equations (18) and (30) we get

$$U = \sum_{\{\alpha_i\},\{\beta_i\}} \tilde{W}(\{-\alpha_i\}, \{-\bar{\beta}_i\}) \prod_{i=1}^{N} D_i(\alpha_i, \bar{\beta}_i)$$
$$\tilde{W}(\{\alpha_i\}, \{\bar{\beta}_i\}) = (2j+1)^{-1} \operatorname{Tr}\left[U \prod_{i=1}^{N} D_i(\alpha_i, \bar{\beta}_i)\right]. \tag{36}$$

This shows again that in our scheme the $\sum[(2j_i+1)^2 - 1]$ displacement operators $D_i(\alpha_i, \bar{\beta}_i)$ are the basic 'building blocks' for general unitary transformations on the large qudits in $H(2j+1)$ (which in fact have $(2j+1)^2 - 1$ generators). The required coefficients are the Weyl functions given above.

In the special case of factorizable transformations $U = \prod U_i$, the Weyl function factorizes

$$\tilde{W}(\{\alpha_i\}, \{\bar{\beta}_i\}) = \prod_{i=1}^{N} \tilde{W}_i(\alpha_i, \bar{\beta}_i) \qquad \tilde{W}_i(\alpha_i, \bar{\beta}_i) = (2j_i+1)^{-1} \operatorname{Tr}[U_i D_i(\alpha_i, \bar{\beta}_i)] \tag{37}$$

and the above equation becomes

$$U = \prod_{i=1}^{N} \left[\sum_{\{\alpha_i\},\{\beta_i\}} \tilde{W}_i(-\alpha_i, -\bar{\beta}_i) D_i(\alpha_i, \bar{\beta}_i)\right]. \tag{38}$$

We can have an analogous scheme in terms of the displacement parity operators. Equations (20) and (33) give

$$U = \sum_{\{\alpha_i\},\{\beta_i\}} W(\{\alpha_i\}, \{\bar{\beta}_i\}) \prod_{i=1}^{N} P_i(\alpha_i, \bar{\beta}_i)$$
$$W(\{\alpha_i\}, \{\bar{\beta}_i\}) = (2j+1)^{-1} \operatorname{Tr}\left[U \prod_{i=1}^{N} P_i(\alpha_i, \bar{\beta}_i)\right]. \tag{39}$$

This shows that the $\sum[(2j_i+1)^2 - 1]$ displaced parity operators $P_i(\alpha_i, \bar{\beta}_i)$ can also be used as the basic 'building blocks' for general unitary transformations on the large qudits in $H(2j+1)$. The required coefficients here are the Wigner functions. Here also, in the special case of factorizable transformations the Wigner function factorizes

$$W(\{\alpha_i\}, \{\bar{\beta}_i\}) = \prod_{i=1}^{N} W_i(\alpha_i, \bar{\beta}_i) \qquad W_i(\alpha_i, \bar{\beta}_i) = (2j_i+1)^{-1} \operatorname{Tr}[U_i P_i(\alpha_i, \bar{\beta}_i)] \tag{40}$$

and the above equation becomes

$$U = \prod_{i=1}^{N} \left[\sum_{\{\alpha_i\},\{\beta_i\}} W_i(\alpha_i, \bar{\beta}_i) P_i(\alpha_i, \bar{\beta}_i)\right]. \tag{41}$$

4. Discussion

We have considered unitary transformations in an angular momentum Hilbert space $H(2j+1)$. The displacement operators are generators of the $SU(2j+1)$ group and finite transformations have been expressed in equation (18) as a finite sum of the displacement operators with the Weyl function as coefficients. An alternative expansion has been given in equation (20) where finite transformations have been expressed as a finite sum of the displaced parity operators with the Wigner function as coefficients. These two expansions show that the displacement operators or the displaced parity operators can be used as 'building blocks' for general unitary transformations of qudits. There are $(2j+1)^2 - 1$ displacement operators (or displaced parity operators) which for large qudits is a large number. A reduction in the number of these building blocks is highly desirable.

The Chinese remainder theorem is used to factorize large qudits in the Hilbert space $H(2j+1)$ in terms of smaller qudits in Hilbert spaces $H(2j_i+1)$. All unitary transformations on large qudits can be performed through appropriate unitary transformations on the smaller qudits. Equation (36) expresses unitary transformations of the large qudits in $H(2j+1)$, as a finite sum of the displacement operators acting on the smaller qudits in the various $H(2j_i+1)$, with the Weyl function as coefficients. There are $\sum[(2j_i+1)^2 - 1]$ such displacement operators; a much smaller number than $(2j+1)^2 - 1$.

Transformations in large Hilbert spaces are complicated and cumbersome and the factorization discussed, simplifies them. Of course we know that if we have a large Hilbert space with dimension $2j+1$ and a product of smaller Hilbert spaces (with $2j+1 = \prod(2j_i+1)$) there exist mappings between them. But the problem is to construct them explicitly and also to find 'intelligent mappings' which are useful in our context. In this paper, we show explicitly in equation (36) how to perform unitary transformations in the large space using the displacement operators in the small spaces. Our scheme requires the $(2j_i+1)$ to be coprime with respect to each other. For example, we can factorize $2j+1$ as product of powers of prime numbers. The scheme does not work when the factors are not coprime, and in this sense it is a bit restrictive. On the other hand this 'mild constraint' leads to strong results like equations (24) and (30). From a practical point of view this restriction is not a major obstacle for an implementation of the scheme.

The work has implications for the recently discussed qudit approach to quantum information processing [1]. It shows that we can use large qudits and perform the necessary unitary transformations through appropriate unitary transformations in smaller qudits, which can be implemented physically more easily.

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